BME 171-02, Signals and Systems

Exam II: Solutions
100 points total

0. (5 pts.) Fourier transform tables.

1. (20 pts.) Determine the Fourier transforms of the following signals:

(a) $x(t) = (\cos(5t) + e^{-2t})u(t)$

Solution:

$$\begin{aligned}
\cos(5t)u(t) &\leftrightarrow \frac{1}{2}\left[\pi\delta(\omega+5) + \frac{1}{j(\omega+5)} + \pi\delta(\omega-5) + \frac{1}{j(\omega-5)}\right] \\
e^{-2t}u(t) &\leftrightarrow \frac{1}{j\omega+2} \\
X(\omega) &= \frac{1}{2}\left[\pi\delta(\omega+5) + \frac{1}{j(\omega+5)} + \pi\delta(\omega-5) + \frac{1}{j(\omega-5)}\right] + \frac{1}{j\omega+2}
\end{aligned}$$

(b) $x(t) = (1-t)\,p_2(t)$

Solution:

$$\begin{aligned}
p_2(t) &\leftrightarrow 2\,\text{sinc}\left(\frac{\omega}{\pi}\right) \\
tp_2(t) &\leftrightarrow 2j\frac{d}{d\omega}\text{sinc}\left(\frac{\omega}{\pi}\right) \\
X(\omega) &= 2\left[\text{sinc}\left(\frac{\omega}{\pi}\right) - j\frac{d}{d\omega}\text{sinc}\left(\frac{\omega}{\pi}\right)\right]
\end{aligned}$$

(c) $x(t) = \int_0^t e^{-3(t-\lambda)}p_1(\lambda - 1/2)d\lambda$

Solution: let $x_1(t) = e^{-3t}u(t)$ and $x_2(t) = p_1(t-1/2)$. Then $x(t) = x_1(t) \star x_2(t)$.

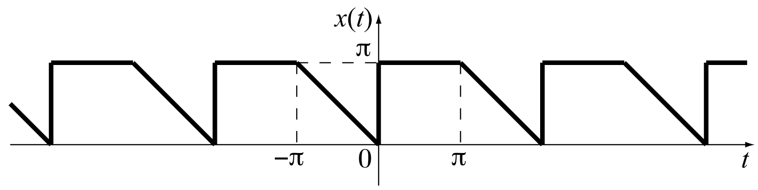
$$\begin{aligned}
x_1(t) &\leftrightarrow \frac{1}{j\omega+3} \\
x_2(t) &\leftrightarrow \text{sinc}\left(\frac{\omega}{2\pi}\right)e^{-j\omega/2} \\
X(\omega) &= X_1(\omega)X_2(\omega) = \frac{1}{j\omega+3}\,\text{sinc}\left(\frac{\omega}{2\pi}\right)e^{-j\omega/2}
\end{aligned}$$

(d) $x(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t \le 2 \\ 2, & t > 2 \end{cases}$

Solution: let $x_1(t) = p_2(t-1)$ and note that $x(t) = \int_0^t x_1(\lambda)d\lambda$. Then

$$\begin{aligned}
x_1(t) &\leftrightarrow 2\,\text{sinc}\left(\frac{\omega}{\pi}\right)e^{-j\omega} \\
X_1(0) &= 2 \\
X(\omega) &= \frac{1}{j\omega}X_1(\omega) + \pi X_1(0)\delta(\omega) = \frac{2}{j\omega}\,\text{sinc}\left(\frac{\omega}{\pi}\right)e^{-j\omega} + 2\pi\delta(\omega)
\end{aligned}$$

2. (25 pts.) Write down the trigonometric Fourier series representation of the following signal:



Solution: we have $T = 2\pi, \omega_0 = \frac{2\pi}{T} = 1$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)dt = \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-t)dt + \int_{0}^{\pi}\pi dt\right) = \frac{1}{2\pi}\left(\left[-\frac{t^2}{2}\right]_{-\pi}^{0} + \pi^2\right) = \frac{1}{2\pi}\left(\frac{\pi^2}{2}+\pi^2\right) = \frac{3\pi}{4}$$

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\cos(kt)dt = \frac{1}{\pi}\left(-\int_{-\pi}^{0} t\cos(kt)dt + \pi\int_{0}^{\pi}\cos(kt)dt\right) \\
&= \frac{1}{\pi}\left(-\left[\frac{1}{k}t\sin(kt)\right]_{-\pi}^{0} + \frac{1}{k}\int_{-\pi}^{0}\sin(kt)dt + \pi\left[\frac{1}{k}\sin(kt)\right]_{0}^{\pi}\right) \\
&= \frac{1}{\pi}\left(0 - \frac{1}{k}\left[\frac{1}{k}\cos(kt)\right]_{-\pi}^{0} + 0\right) = \frac{\cos(\pi k) - 1}{\pi k^2} \\
&= \begin{cases} 0, & \text{if } k \text{ is even} \\ -\frac{2}{\pi k^2}, & \text{if } k \text{ is odd} \end{cases}
\end{aligned}$$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\sin(kt)dt = \frac{1}{\pi}\left(-\int_{-\pi}^{0} t\sin(kt)dt + \pi\int_{0}^{\pi}\sin(kt)dt\right) \\
&= \frac{1}{\pi}\left(\left[\frac{1}{k}t\cos(kt)\right]_{-\pi}^{0} - \frac{1}{k}\int_{-\pi}^{0}\cos(kt)dt - \pi\left[\frac{1}{k}\cos(kt)\right]_{0}^{\pi}\right) \\
&= \frac{1}{\pi}\left(\frac{1}{k}\pi\cos(\pi k) + 0 - \pi\left(\frac{1}{k}\cos(k\pi) - \frac{1}{k}\right)\right) \\
&= \frac{1}{k}
\end{aligned}$$

Thus,

$$x(t) = \frac{3\pi}{4} - \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \frac{2}{\pi k^2}\cos(kt) + \sum_{k=1}^{\infty}\frac{1}{k}\sin(kt)$$

3. (20 pts.) An LTI system generates the output

$$y(t) = (e^{-2t} - e^{-3t})u(t)$$

in response to the input $x(t) = e^{-2t}u(t)$.

(a) Determine the unit impulse response $h(t)$ of the system.

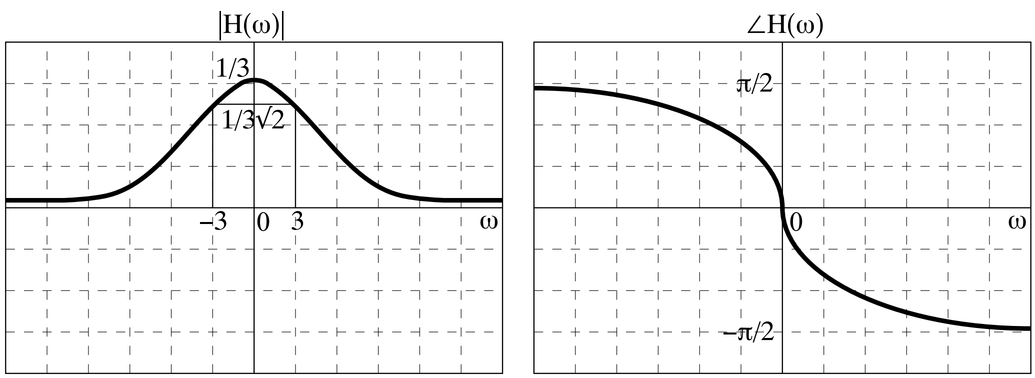
Solution: since the system is LTI, $y(t) = x(t) \star h(t)$. In the frequency domain,

$$\begin{aligned}
Y(\omega) &= X(\omega)H(\omega); \\
X(\omega) &= \frac{1}{j\omega + 2} \\
Y(\omega) &= \frac{1}{j\omega+2} - \frac{1}{j\omega+3} = \frac{1}{(j\omega+2)(j\omega+3)} \\
H(\omega) &= \frac{Y(\omega)}{X(\omega)} = \frac{j\omega+2}{(j\omega+2)(j\omega+3)} = \frac{1}{j\omega+3}.
\end{aligned}$$

Therefore,

$$h(t) = e^{-3t}u(t).$$

(b) Sketch the amplitude $|H(\omega)|$ and the phase $\angle H(\omega)$ in the sets of axes provided. Be sure to mark the axes properly.



$$\begin{aligned}
H(\omega) &= \frac{1}{j\omega+3} \\
&= \frac{3}{j\omega+3} - j\frac{\omega}{\omega+3}; \\
|H(\omega)| &= \frac{1}{\sqrt{\omega^2+9}} \\
\angle H(\omega) &= \tan^{-1}\left(-\frac{\omega}{3}\right) \\
|H(0)| &= \frac{1}{3}; \\
|H(\pm 3)| &= \frac{1}{\sqrt{2}}|H(0)| = \frac{1}{3\sqrt{2}}
\end{aligned}$$

4. (20 pts.) Consider the discrete-time signal

$$x[n] = \cos(3n)p[n],$$

where $p[n]$ is the rectangular pulse

$$p[n] = \begin{cases} 1, & n = 0, 1, \ldots, 6 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute its discrete-time Fourier transform (DTFT) $X(\Omega)$.

Solution:

$$\begin{aligned} P(\Omega) &= \frac{\sin(7\Omega/2)}{\sin(\Omega/2)} e^{-j3\Omega} \\ X(\Omega) &= \frac{1}{2}\left[P(\Omega+3) + P(\Omega-3)\right] \\ &= \frac{1}{2}\left[\frac{\sin(7(\Omega+3)/2)}{\sin((\Omega+3)/2)} e^{-j3(\Omega+3)} + \frac{\sin(7(\Omega-3)/2)}{\sin((\Omega-3)/2)} e^{-j3(\Omega-3)}\right] \end{aligned}$$

(b) Express the 7-point discrete Fourier transform (DFT) of $x[n]$ in terms of the DTFT $X(\Omega)$.

Solution: for $k = 0, 1, \ldots, 6$,

$$\begin{aligned} X_k &= X(2\pi k/7) \\ &= \frac{1}{2}\left[\frac{\sin(7(2\pi k/7+3)/2)}{\sin((2\pi k/7+3)/2)} e^{-j3(2\pi k/7+3)} + \frac{\sin(7(2\pi k/7-3)/2)}{\sin((2\pi k/7-3)/2)} e^{-j3(2\pi k/7-3)}\right] \end{aligned}$$

5. (10 pts.) You have two discrete-time signals, $x[n]$ and $\nu[n]$, where $x[n] = 0$ for $n < 0$ and $n \geq 1000$ and $\nu[n] = 0$ for $n < 0$ and $n \geq 1040$. Explain how you would use the FFT algorithm in order to efficiently compute the convolution $x[n] \star \nu[n]$ and estimate the number of (complex) multiplications you would need.

   Solution: in general, the convolution of $x[n]$ and $\nu[n]$ will have $1000 + 1040 = 2040$ nonzero components. The smallest power of 2 that is larger than 2040 is $L = 2048 = 2^{11}$. Let us pad $x[n]$ and $\nu[n]$ with zeros so that

$$
\begin{aligned}
x[n] &= 0, & n &= 1000, 1001, \ldots, 2048 \\
\nu[n] &= 0, & n &= 1040, 1041, \ldots, 2048
\end{aligned}
$$

   Then to compute $x[n] \star \nu[n]$, we would first use the FFT algorithm to compute the L-point DFT's X_k and V_k of $x[n]$ and $\nu[n]$, and then use the FFT algorithm to compute the inverse L-point DFT of the product of X_k and V_k.

   We will need:

   - On the order of $(1/2)L\log_2 L = (1/2) \cdot 2048 \cdot 11 = 11264$ multiplications to compute the L-point DFT of $x[n]$.
   - On the order of $(1/2)L\log_2 L = (1/2) \cdot 2048 \cdot 11 = 11264$ multiplications to compute the L-point DFT of $\nu[n]$.
   - $L = 2048$ multiplications to compute the product of X_k and V_k.
   - On the order of $(1/2)L\log_2 L = (1/2) \cdot 2048 \cdot 11 = 11264$ multiplications to compute the L-point inverse DFT of the product of X_k and V_k.

   Thus, the total number of multiplications is on the order of

$$
11264 + 11264 + 2048 + 11264 = 35840.
$$